Probability of Error for Quadratic Detectors

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A procedure is presented for evaluating the performance of a general class of digital detectors which square or multiply signal waves contaminated by gaussian noise. In addition to simplifying and unifying the treatment of a number of previously solved problems and some hitherto unsolved ones, the method achieves a considerable advance toward a complete evaluation of postdetection filtering. In contrast to most of the earlier related work, which is typically restricted to filters described as accepting low-frequency difference products and rejecting high-frequency sum products, the present analysis offers a tractable inclusion of filters which do significant selective processing of the detected low-frequency signal and noise components.

A principal goal sought is the asymptotic form approached by the error probability expressed as a function of the signal-to-noise ratio when the latter is large. This is a primary region of interest in digital data transmission over the telephone network, and the applicable results give a basis for comparing performance of different systems. The mathematical problem is one of calculating the probability that a quadratic form in a set of gaussian variables with arbitrary means and variances will assume values critically far removed from that obtained when each variable is at its mean. The mean values represent signal contributions unperturbed by noise and for good performance should dominate over the noise except at the tails of the distribution. Concentration of attention on the infrequent large noise peaks calls for an approach inherently different from the conventional series expansions appropriate near the center of the distribution. The results are of importance not only in detection theory but also in general statistical analysis of rare events.

I. INTRODUCTION

A general class of data receivers have decision logic based on observing at the output the sign of a quadratic form

$$q = \sum_{i,j=1}^{n} Q_{ij} w_i w_j \equiv w^+ Q w. \tag{1}$$

Examples of such quadratic detectors and the reduction of their output to the form (1) are given in Sections IV and V, and a schematic representation in Fig. 1. Let it suffice for the present to say that the real symmetric matrix Q is determined by the system filters, while the real N-dimensional vector w and its transpose w^+ are related to the received signal plus noise.

During a considerable portion of this paper we shall be concerned with the asymptotic evaluation of the probability of error for such receivers when the noise is additive gaussian. To develop the tools

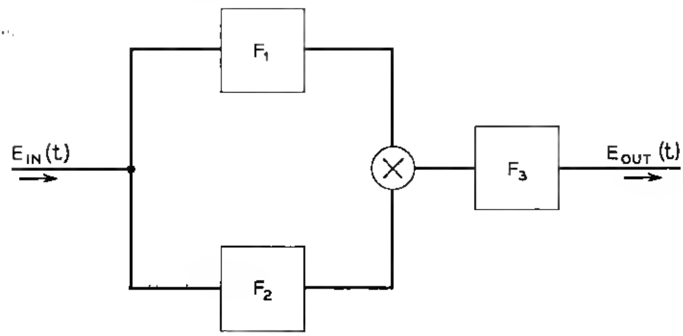

Fig. 1 — Model of the quadratic detector.

for this end, and also partly for general mathematical interest, Sections II and III are devoted to the following general problem: Given that the real gaussian vector w whose components have means $\bar{w}_i$ and a positive definite* covariance matrix M,

$$M_{ij} = \langle (w_i - \bar{w}_i)\ (w_j - \bar{w}_j) \rangle = M_{ji}, \tag{2}$$

is such that the quadratic form $\bar{w}^+ Q \bar{w}$ is positive, what is the probability P_e that for small noise (i.e., that the w_i have small variances) the quadratic form (1) is negative? We refer to P_e as the asymptotic probability of error.† Various special cases of this problem have been treated in the literature; some references may be found in Section IV where we reproduce and generalize some of these special results.

* Without any real loss of generality strict positive definiteness is assumed.

† As pointed out later our techniques may also be used to obtain the distribution function in certain regions, not only the probability of error.

A systematic treatment of the general problem has hitherto been lacking. Progress has been inhibited due to the fact that the exact probability distribution of arbitrary gaussian quadratic forms is too complicated to be useful. Experience has shown that the curves for the probability of error vs signal-to-noise ratio tend to be roughly parallel for different data receivers and to be characterized sufficiently well by their asymptotic slopes. We, therefore, do not strive to obtain exact error rates but rather deal with asymptotic forms for large signal-to-noise ratios. In other words, we will be concerned with the behavior of the distribution on the tails of the density functions of our quadratic forms. For high error rates or low signal-to-noise ratios, other approximations may be obtained by using various well-known moment series such as the Edgeworth and Gram-Charlier expansions.

We show in Section V how our results may be used to attack the problem of the distribution of the filtered response of a product detector. It is primarily this consideration of postdetection filtering that has been absent from earlier discussions. Finally in Section VI, a simple model of a fairly representative class of quadratic detectors is analyzed in some detail by means of a rapidly convergent expansion in prolate spheroidal wave functions.

II. GENERAL ANALYSIS FOR QUADRATIC FORM

In the introduction we defined the problem of obtaining the probability of error for the quadratic form q. We approach this problem via the characteristic function $C(\omega)$ of (1), which is well known to be given by[1]

$$C(\omega) \equiv \langle e^{i\omega q}\rangle_q = \frac{\exp \frac{1}{2}[\bar{w}^+M^{-1}(I - 2i\omega MQ)^{-1}\bar{w} - \bar{w}^+M^{-1}\bar{w}]}{\sqrt{\det (I - 2i\omega MQ}}. \tag{3}$$

The symbol $\langle\cdot\rangle_q$ denotes the average with respect to q; "det" means determinant and I is the identity matrix. Since M^{-1} is the inverse of a real symmetric and positive definite matrix, it itself has all of these properties. We now note the following theorem:[2] Let A and B be real symmetric matrices, and further let A be positive definite. Then there exists a real matrix S such that

$$S^+AS = I \tag{4}$$

$$S^+BS = \sigma^2D, \tag{5}$$

where D is some diagonal matrix and σ^2 is a positive parameter introduced for later convenience.

Equation (4) implies, in particular, that S^{-1} exists. If we identify A with M^{-1}, and B with Q, (4) and (5) tell us that we may write

$$M = SS^{+} \tag{4a}$$

$$Q = (S^{+})^{-1}\sigma^2 DS^{-1}. \tag{5a}$$

If we substitute (4a) and (5a) into (3), and further introduce a new real gaussian vector y by the linear transformation

$$y = S^{-1}w, \tag{6}$$

we arrive at a simpler expression for the characteristic function, namely,

$$C(\omega) = \frac{\exp\left[\frac{1}{2}\bar{y}^{+}(I - 2i\omega\sigma^2 D)^{-1}\bar{y} - \frac{1}{2}\bar{y}^{+}\bar{y}\right]}{\sqrt{\det(I - 2i\omega\sigma^2 D)}}. \tag{7}$$

The principal simplification is now that D is a diagonal matrix. We note that

$$w^{+}Qw = \sigma^2 y^{+}Dy \tag{8}$$

and

$$\bar{w}^{+}Q\bar{w} = \sigma^2 \bar{y}^{+}D\bar{y}. \tag{9}$$

We find it more convenient to deal with q when it is expressed in terms of the variables y_i.

The probability of error defined earlier may be expressed in terms of the characteristic function by the formula

$$\Pr\{q < 0\} \equiv P_e = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{C(\omega)}{\omega + i\epsilon}\, d\omega. \tag{10}$$

The $i\epsilon$ ($\epsilon > 0$) appearing in the denominator of the integrand and in (10) is used to signify that in the complex ω-plane, the contour of integration implied in (10) goes above the singularity at $\omega = 0$. Making use of (7), we write (10) in more detail*

$$P_e = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\omega}{\omega + i\epsilon} \frac{1}{\sqrt{\prod_{j=1}^{N}(1 - 2i\omega\sigma^2 d_j)}} \cdot \exp\left[\sum_{j=1}^{N} \frac{i\omega d_j y_j^2}{1 - 2i\omega\sigma^2 d_j}\right] \tag{11}$$

* Henceforth, we will not use bars to denote the mean of a random variable.

$$= -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\omega}{\omega + i\epsilon}\,\frac{1}{\sqrt{\prod_{j=1}^{N}(1 - i\omega d_j)}} \cdot \exp\left[-\frac{\omega}{2\sigma^2}\sum_{j=1}^{N}\frac{y_j^2}{\omega + \dfrac{i}{d_j}}\right], \tag{12}$$

where d_j is the jth element of the real diagonal matrix D. Equation (12) follows from (11) by a simple change of variable. We observe in (12) that the quantity $1/\sigma^2$ appears only in the exponent, and therefore we may obtain an asymptotic expansion of P_e valid for small σ^2 by considering only the exponent. One will see in later sections that this asymptotic result for small σ^2 corresponds to the asymptotic result for large signal-to-noise ratios.

Proceeding with the analysis of (12), we remark that the integrand obviously falls off sufficiently rapidly at infinity to allow one to close or distort the contour at infinity without changing the value of the integral. We further note the singularities of the integrand of (12). There is a simple pole at $\omega = 0$ which has already been discussed. In addition to the simple pole at $\omega = 0$, the exponent has simple poles at $\omega = -i/d_i$; these all lie along the imaginary axis. Also at these points the denominator of the integrand has, in general, branch points due to the square root. For doubly degenerate eigenvalues, the branch points become simple poles.

We now concentrate our attention on performing the integration in (12) for small σ^2 by the saddle point method.[3,4] To locate the saddle points, let $\omega = i\eta$, and consider the solutions to

$$-\frac{d}{d\eta}\left[\eta\sum_{i=1}^{N}\frac{y_i^2}{\eta + \dfrac{1}{d_i}}\right] = 0, \tag{13}$$

or

$$F(\eta) \equiv -\sum_{i=1}^{N}\frac{\dfrac{y_i^2}{d_i}}{\left(\eta + \dfrac{1}{d_i}\right)^2} = 0. \tag{14}$$

Consider $F(\eta)$ defined by (14) and let η be real. We shall show that there always exists a saddle point for positive η (ω positive imaginary),

which occurs *before* the first singularity of the exponent on the positive imaginary axis.* We separate the sum (14) into two parts

$$F(n) = -\sum_{d_i \text{pos}} \frac{\dfrac{y_i^2}{d_i}}{\left(n + \dfrac{1}{d_i}\right)^2} - \sum_{d_i \text{neg}} \frac{\dfrac{y_i^2}{d_i}}{\left(n + \dfrac{1}{d_i}\right)^2}. \tag{15}$$

We note that when $n \approx -(1/d_i)$, $F(n)$ is large and positive for $d_i < 0$ and $F(n)$ is large and negative for $d_i > 0$. Also note that $F(0) = -\sum d_i y_i^2 \leqq 0$ by (9) and our assumption that the noise-free signal is positive. Hence by continuity, there must be at least one $n > 0$ for which $F(n) = 0$, which locates a saddle point for us. Clearly by the monotonicity of the two parts of expression (15) there can be only one such saddle point between the origin and the first singularity of the exponent on the positive imaginary axis.

The relations in the complex ω-plane described above are illustrated in Fig. 2. The dotted line labeled L_1 is the original contour of integration, and the small circle labeled "*S.P.*" is the (unique) saddle point on the positive imaginary axis lying before the nearest singularity $\omega = -i/d_i$, $d_i < 0$, which has nonvanishing residue y_i^2 in the exponent of the integrand. Also drawn on the imaginary axis in Fig. 2 are the branch cuts of the denominator, indicated by the hatched bars in the figure. We assume in this section (and Fig. 2 is so drawn) that the saddle point occurs before the nearest singularity $\omega = -i/d'$ on the positive imaginary axis.† As discussed above, this is guaranteed to be the case if only $(y')^2 = 0$.‡

If the saddle point is situated as shown in Fig. 2, the contour may then be shifted from the real axis (line L_1 in Fig. 2) to a contour which is a straight line parallel to the real axis and passing through the saddle point (the line L_2 in Fig. 2). The integrand drops off sufficiently rapidly for large $|\omega|$ so that the ends of the contour connecting L_1 and L_2 (in accordance with Cauchy's theorem) give no contribution. We will now, for large signal-to-noise ratios (small σ^2), approximate the integral along the contour L_2 by the contribution in the immediate neighborhood of the saddle point. It is shown in the appendix that the magnitude

* This assumes that at least one of the d_i is negative with nonvanishing residue y_i^2. We also, of course, are assuming that the quadratic form q in the absence of noise is positive.

† For definiteness we have based our analysis on the assumption that the noiseless signal is positive. We emphasize that the role of upper and lower half planes would be interchanged if one assumed that the noiseless signal were negative.

‡ We have used d' to denote that particular d_i which corresponds to the nearest singularity on the positive imaginary axis; y' is the associated y_i.

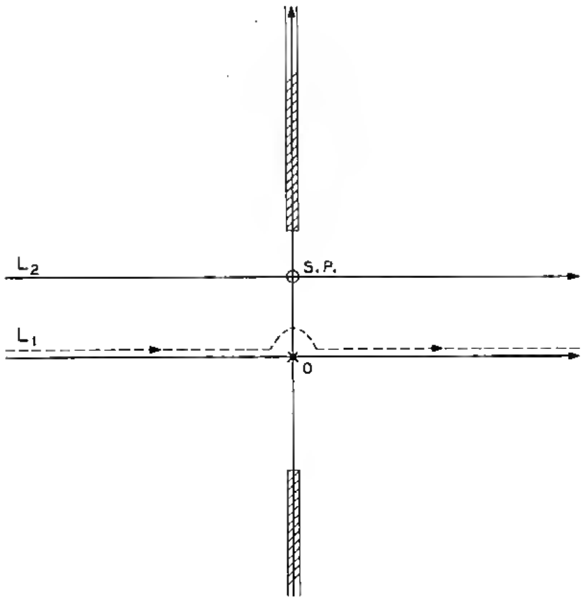

Fig. 2 — Contours and singularities in the ω-plane.

of the exponential term in the integrand of (12) is a monotonically decreasing function of ω as one recedes from the saddle point on the contour L_2, and therefore this saddle point evaluation is asymptotically correct.[3,4] One might also add that it can be shown that the contour L_2 is in fact tangent to the path of steepest descent at the saddle point.

To obtain an explicit formula for our asymptotic evaluation of P_e under these conditions, write (12) as

$$P_e = -\frac{1}{2\pi i}\int_{L_2} g(z)\,\exp -\left[\frac{f(z)}{\sigma^2}\right] dz, \qquad (16)$$

with

$$g(z) = \frac{1}{z}\,\frac{1}{\sqrt{\prod(1 - izd_i)}} \qquad (17)$$

and

$$f(z) = \frac{z}{2}\sum_{j=1}^{N}\frac{y_j^2}{z + \dfrac{i}{d_j}}. \qquad (18)$$

Then, by a standard saddle point evaluation, we have

$$P_e \sim \frac{1}{\sqrt{2\pi}} \frac{1}{\Gamma\sqrt{\prod(1+\Gamma d_j)}} \frac{\sigma}{\sqrt{f''(i\Gamma)}} \exp\left[-\frac{\Gamma}{2\sigma^2}\sum \frac{y_j^2}{\Gamma + \dfrac{1}{d_j}}\right], \tag{19}$$

where, to repeat, Γ is the smallest positive root of the equation

$$\frac{d}{dz} f(z) \,|_{z=i\Gamma} = 0. \tag{20}$$

The notation $f''(i\Gamma)$ in (19) means the second derivative of $f(z)$ evaluated at $z = i\Gamma$.

We finally wish to note that although we have concentrated in this section on the evaluation of P_e, one could use entirely analogous techniques to find an asymptotic expression for the probability P_K that the quadratic form q is less than some number K as long as K is less than the value of q in the absence of noise. The analog of (10) is (for any value of K)

$$P_K = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{C(\omega)}{\omega + i\epsilon} e^{-i\omega K} d\omega. \tag{21}$$

For a K satisfying the stated condition, the integrand in (21) possesses a saddle point such that a discussion for asymptotically calculating P_K may be given which is entirely analogous to that already given for P_e. For a given σ^2, the accuracy of such an approach will depend on K.

III. VANISHING OF CRITICAL RESIDUE IN UPPER HALF-PLANE

We wish, in this section, to review a particularly trivial example to illustrate a violation of a condition necessary for the guaranteed applicability of our method, namely the nonvanishing of the residue associated with the nearest pole in the upper half-plane. In the example we consider here, the exact answer can be obtained by a simple contour integral. Consider the quadratic form

$$z_1 = x_1^2 - x_2^2 + x_3^2 - x_4^2, \tag{22}$$

where the x's are independent gaussian variables, all with the same variance σ^2. Further, assume that x_2 and x_4 have zero means. Then the characteristic function of z is

$$C(\omega) = \frac{\exp i\omega \dfrac{x_1^2 + x_3^2}{1 - 2i\omega\sigma^2}}{(1 - 2i\omega\sigma^2)(1 + 2i\omega\sigma^2)}. \tag{23}$$

The contour integral implied in (10) may be closed in tbe upper half-plane to yield immediately by exact methods

$$P_e = \tfrac{1}{2}\exp\left[-\frac{x_1^2 + x_3^2}{4\sigma^2}\right]. \tag{24}$$

The exponent in (24) is simply obtained by evaluating the exponent in (23) at the singularity $\omega = i/2\sigma^2$ in the upper half-plane. Now consider instead the expression

$$z_2 = z_1 + \sum_{n=2}^{N} x_{2n+1}^2 - \sum_{n=2}^{N} x_{2n}^2, \tag{25}$$

where z_2 in the absence of noise is assumed positive, the additional gaussian variables are assumed to have variances which are less than σ^2, and the variable x_{2n} $(n \geqq 3)$ with smallest variance has nonvanishing mean. The exponent $[-f(iy)/\sigma^2]$ for the characteristic function of z_2 as one travels up the positive imaginary axis appears as in Fig. 3, where we have written $\omega = iy$. Two situations are clearly possible. First $y_0 < 1/2\sigma^2$; in this case the contour of integration for (10) may be distorted to pass through the saddle point, and the previous discussion then applies. Since $x_2^2 + x_4^2 = 0$ this could not be guaranteed *a priori*. The second case is $y_0 > 1/2\sigma^2$; in order to distort the contour to pass through the saddle point in this situation, one must first sweep the contour past the singularity at $\omega = i/2\sigma^2$. Since this singularity is, for the present case, only a simple pole, the result of pushing the contour past is simply to pick up the residue of this pole. As an asymptotic answer one then has this residue plus the saddle point contribution. However, it is clear from Fig. 3 that the residue term (recall $y_0 > 1/2\sigma^2$) has a less negative

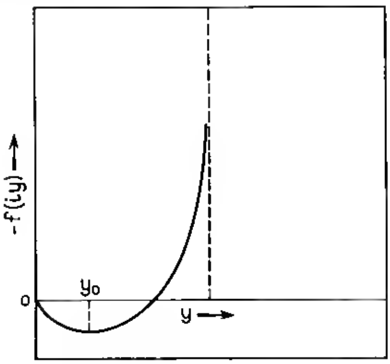

Fig. 3 — Behavior of exponent discussed in section III.

exponent, and thus in the limit of large signal-to-noise ratio will be exponentially dominant over the saddle point. If, for this second case, one has instead one or more branch points before the saddle point, the analysis is not as simple, but realizing that one may distort up to the first singularity (which, of course, has vanishing residue in the exponent), and keeping in mind Fig. 3, one might argue that the exponential behavior for this case is determined by the value of the exponent evaluated at the nearest singularity in the upper half-plane.

IV. SIMPLE COMMUNICATION APPLICATION

In this section we apply our saddle point technique to a number of problems whose asymptotic forms have appeared previously in the literature with derivations based on techniques different from ours. The reproduction of previous results helps to establish confidence in our methods, and arrives at these answers in a more straightforward manner than previously.

The problems considered here may all be put into the form where

$$q = u_1^2 + u_2^2 - v_1^2 - v_2^2 \equiv u^2 - v^2. \tag{26}$$

That is, q is the difference of the squared lengths of two two-dimensional gaussian vectors. We let the variance of both components of the vector $\mathbf{u}$ be equal to σ_2^2 and of both those of $\mathbf{v}$ equal to σ_1^2. Further, all components are independent. We shall see later that analysis of binary or multilevel FM using discrimination detection and differential detection of FM[9] reduces to this case with, in general, $\sigma_1 \neq \sigma_2$, while the analysis for differential phase modulation is also of this form, with $\sigma_1 = \sigma_2$.

The characteristic function for q defined by (26) is simply

$$C(\omega) = \frac{\exp i\omega\left[\dfrac{u^2}{1 - 2i\omega\sigma_2^2} - \dfrac{v^2}{1 + 2i\omega\sigma_1^2}\right]}{(1 - 2i\omega\sigma_2^2)(1 + 2i\omega\sigma_1^2)}, \tag{27}$$

and the probability of error is

$$P_e = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\omega}{\omega + i\epsilon}\,\frac{\exp -\dfrac{\omega}{2\sigma_1^2}\left[\dfrac{v^2}{\omega - iA} + \dfrac{u^2}{A^2\left(\omega + \dfrac{i}{A}\right)}\right]}{(\omega - iA)\left(\omega + \dfrac{i}{A}\right)}, \tag{28}$$

where

$$A = \sigma_2/\sigma_1. \tag{29}$$

Letting, again, $\omega = iy$, we find that the saddle point of interest is at

$$y_0 = A \frac{u^2 + A^2v^2 - uv(1 + A)^2}{u^2 - A^4v^2}. \tag{30}$$

Using (19), we then obtain for the probability of error*

$$P_e \sim \frac{1}{\sqrt{2\pi}} \frac{\sigma_1}{\sqrt{1 + A^2}} \frac{1}{\sqrt{uv}} \frac{(u - A^2v)(u + A^2v)}{[u^2 + A^2v^2 - uv(1 + A^2)]} \cdot \exp\left[\frac{(u - v)^2}{2\sigma_1^2(1 + A^2)}\right]. \tag{31}$$

We might note in passing that if one has two independent N-dimensional gaussian vectors $\mathbf{u}_N$ and $\mathbf{v}_N$ and if all the components of $\mathbf{u}_N$ ($\mathbf{v}_N$) are independent and have the same variance $\sigma_2^2(\sigma_1^2)$, then the probability of error for the difference of the square of these vectors, $u_N^2 - v_N^2$, will have different multiplicative factors from these in (31) but the exponent in (31) will still be correct when u and v are reinterpreted to be the lengths of the vectors $\mathbf{u}_N$ and $\mathbf{v}_N$, respectively. The result (31) and its generalizations for higher dimensions may also be viewed as giving the asymptotic form of the cumulative probability distribution for the doubly noncentral F-distribution. Exact, but not very transparent, formulas for this problem have recently been published by Price.[5]

Analysis of the error performance of binary FM and differential phase modulation leads one to consider the probability that the inner product q of two independent 2-dimensional gaussian vectors, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$,

$$q = 2\boldsymbol{\alpha}\cdot\boldsymbol{\beta} \tag{32}$$

is negative when the inner product of the means is positive.[6,7,8] Here again the components of each vector are independent, and those of $\boldsymbol{\alpha}$ have the same variance and those of $\boldsymbol{\beta}$ have possibly a different variance. However, it is clear that multiplication of (32) by a positive constant can adjust these two variances to be identical (with appropriate adjustment of the means) without affecting the probability of error. Hence, it suffices to choose the variances to be equal to the same constant σ^2. Further, introduce

$$\mathbf{u} = (\boldsymbol{\alpha} + \boldsymbol{\beta})/\sqrt{2}$$
$$\mathbf{v} = (\boldsymbol{\alpha} - \boldsymbol{\beta})/\sqrt{2}, \tag{33}$$

so that

* If we write (31) as $P_e = f(u,v,\sigma_1,\sigma_2)$, then, if the noisefree form (26) is *negative*, the probability P_e' that q is positive is $P_e' = -f(u,v,\sigma_1,\sigma_2)$.

$$q = u^2 - v^2. \tag{34}$$

The conditions assumed in the derivation of (31) are satisfied by (34), and in particular we have

$$\sigma_1^2 = \sigma_2^2 = \sigma^2. \tag{35}$$

Equation (31) for the probability of error then simplifies to

$$P_e \sim \frac{\sigma}{2\sqrt{\pi}} \frac{1}{\sqrt{uv}} \frac{u+v}{u-v} \exp\left[-\frac{(u-v)^2}{4\sigma^2}\right]. \tag{36}$$

By making extensive use of (33), it is readily verified that our expression (36) for the asymptotic (large S/N) probability of error for q defined by (32) is identical with the expression given in Bennett and Davey,[6] (9–56). The considerably more complicated form of the Bennett-Davey result is solely due to the fact that they express their answer in terms of variables $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ instead of u and v.

We now consider another application of our formula (31), namely to the analysis of errors in multilevel FM data transmission using discrimination detection. In this application, one is concerned with the probability that the instantaneous frequency ψ is in error, where ψ is given in terms of in-phase and quadrature components, $x(t)$ and $y(t)$ respectively, by the equation[10,11]

$$\psi = \frac{x\dot{y} - y\dot{x}}{x^2 + y^2}. \tag{37}$$

The quantities x, y, $\dot{x}$, $\dot{y}$ are gaussian variables with arbitrary means, and with variances equal to σ^2, σ^2, $\dot{\sigma}^2$, $\dot{\sigma}^2$ respectively. We also use the notation

$$\begin{aligned} R^2 &= x^2 + y^2 \\ R\dot{R} &= x\dot{x} + y\dot{y}. \end{aligned} \tag{38}$$

Suppose for the noise-free situation $\psi > z$. It is of interest in the multilevel situation to know the probability P that $\psi < z$ in the presence of gaussian noise. To put this problem into a form to which (31) is directly applicable, we made use of the following chain of equalities:

$$\begin{aligned} P &= \Pr[\psi \leqq z] = \Pr[\psi - z \leqq 0] \\ &= \Pr[x(\dot{y} - zx) + y(-\dot{x} - zy) \leqq 0] \\ &= \Pr[xa + yb \leqq 0], \end{aligned}$$

where

$$\begin{aligned} a &= k(\dot{y} - zx) \\ b &= -k(\dot{x} + zy), \end{aligned} \tag{39}$$

and k is any *positive* constant.

We define

$$\begin{aligned} u_1 &= x + a \\ u_2 &= y + b \\ v_1 &= x - a \\ v_2 &= y - b \end{aligned} \tag{40}$$

and further choose k so that

$$k^2 = \frac{1}{z^2 + \rho^2}, \tag{41}$$

where

$$\rho^2 = \frac{\dot{\sigma}^2}{\sigma^2}. \tag{42}$$

Then we see that

$$P = \Pr\,(u_1^2 + u_2^2 - v_1^2 - v_2^2 \leqq 0), \tag{43}$$

where the set of variables (u_1, u_2, v_1, v_2) are all independent (due to this choice of k) and further,

$$\sigma_{u_1}^2 = \sigma_{u_2}^2 = \sigma^2(1 - kz)^2 + k^2\dot{\sigma}^2 = \sigma_2^2 \tag{44a}$$

$$\sigma_{v_1}^2 = \sigma_{v_2}^2 = \sigma^2(1 + kz)^2 + k^2\dot{\sigma}^2 = \sigma_1^2. \tag{44b}$$

It is useful to write $u^2 = u_1^2 + u_2^2$ and $v^2 = v_1^2 + v_2^2$ in terms of the original FM variables. Using (37) – (40), we find that

$$u^2 = R^2\left\{\left[1 + kz\left(\frac{\dot{\psi}}{z} - 1\right)\right]^2 + k^2\left(\frac{\dot{R}}{R}\right)^2\right\} \tag{45a}$$

$$v^2 = R^2\left\{\left[1 - kz\left(\frac{\dot{\psi}}{z} - 1\right)\right]^2 + k^2\left(\frac{\dot{R}}{R}\right)^2\right\}. \tag{45b}$$

A convenient simplification results if we restrict ourselves to constant amplitude FM waves, i.e., $\dot{R} = 0$. For this specialization we have

$$u = R[1 + k(\dot{\psi} - z)] \tag{46a}$$

$$v = R\begin{cases} 1 - k(\dot{\psi} - z) & \text{if } k(\dot{\psi} - z) < 1 \\ k(\dot{\psi} - z) - 1 & \text{if } k(\dot{\psi} - z) > 1. \end{cases} \tag{46b}$$

If we set $z = 0$, we immediately have that $k = \sigma/\dot{\sigma}$, $\sigma_1^2 = \sigma_2^2 = 2\sigma^2$, and, distinguishing the two cases in (46b), (46) and (31) reproduce exactly (38) and (39) of Ref. (6).

Equations (46) and (31) together provide general formulas for the asymptotic evaluation of multilevel FM.

V. GENERAL QUADRATIC DETECTOR

In this section we consider the probability of error for the filtered response of the product detector given in Fig. 1 when the noise input is white gaussian. The signal $s(t)$ plus the added noise $n(t)$ is divided into two branches, each of which has a filter (denoted by F_1 and F_2 for the two branches). The outputs of these filters are multiplied and the product is passed through the filter F_3, whose output is the final system output. This product detector is a generalization of the square-law detector considered by Kac and Siegert,[12] and by Emerson.[13] We first parallel Emerson's treatment and express the problem as in expression (1), except now $N = \infty$.

One familiar with Refs. 12 and 13 will not be surprised that the results amount to solving an integral equation, one which there is little hope of solving in practice. Therefore, we feel that an important point is made when we show in the next section that for a particular model of considerable practical interest, one can effectively approximate the system function by a kernel of finite rank, expressible in terms of known functions; the functions we have in mind for this purpose are the prolate spheroidal wave functions.[1] We wish to emphasize that although the results of earlier sections are applied here, we regard the rapidly converging approximations to the system function as important in the treatment of this problem.

Following Emerson, let $f_i(t)$ be the impulse response of the ith filter. Then the output at time t, $E_{\text{out}}(t)$, may be written in terms of the input wave

$$E_{\text{in}}(t) = s(t) + n(t) \tag{47}$$

as

$$E_{\text{out}}(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} du\, dv\, E_{\text{in}}(t-u)g(u,v)E_{\text{in}}(t-v), \tag{48}$$

where the system function $g(u,v)$ is given by

$$g(u,v) = \int_{-\infty}^{\infty} f_1(u-z)f_3(z)f_2(v-z)dz. \tag{49}$$

One can immediately see from (49) that if the filters F_1 and F_2 are not identical, then $g(u,v)$ is not a symmetric function of u and v. How-

ever, it is apparent from (48) that if we write the identity

$$g(u,v) = \tfrac{1}{2}[g(u,v) + g(v,u)] + \tfrac{1}{2}[g(u,v) - g(v,u)], \tag{50}$$

then when (50) is used in (48), the second term in (50) gives no contribution, and we have

$$E_{\text{out}}(t) = \iint du\, dv\, E_{\text{in}}(t-u)G(u,v)E_{\text{in}}(t-v), \tag{51}$$

where

$$G(u,v) = \tfrac{1}{2}[g(u,v) + g(v,u)]. \tag{52}$$

The kernel $G(u,v)$ is now hermitian, and all its eigenvalues are guaranteed to be real; we also assume that G is square integrable. If λ_n denotes its nth eigenvalue and $\varphi_n(t)$ its nth eigenfunction, then we may write the well-known operator result*

$$G(u,v) = \Sigma\lambda_n\varphi_n(u)\varphi_n(v). \tag{53}$$

Thus,

$$E_{\text{out}}(t) = \Sigma\lambda_n e_n^2(t), \tag{54}$$

where

$$e_n(t) = \int_{-\infty}^{\infty} \varphi_n(v)E_{\text{in}}(t-v)\, dv\,. \tag{55}$$

Upon using (47) and the fact that $n(t)$ represents white gaussian noise with correlation function

$$\langle n(t)n(t')\rangle = N_0\delta(t-t'), \tag{56}$$

and suppressing the t-dependence in (55), we see that the e_n are independent gaussian variables with variances given by

$$\text{var}\,\{e_n\} = N_0\,. \tag{57}$$

The probability P_e may now, in accordance with Section II, be written as

$$P_e = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\omega}{\omega+i\epsilon}\prod\frac{1}{\sqrt{1-i\omega\lambda_n}} \cdot\exp\left[-\frac{1}{2N_0}\sum_n\frac{\omega e_n^2}{\omega+i/\lambda_n}\right], \tag{58}$$

* We assume without loss of generality that the $\varphi_n(t)$ are real.

or

$$P_e \sim \frac{\sqrt{2N_0}}{\sqrt{2\pi}} \frac{1}{y_0} \frac{1}{\sqrt{\prod(1+y_0\lambda_n)}} \frac{\exp\left[-\frac{1}{2N_0}f(y_0)\right]}{\sqrt{-f''(y_0)}}. \tag{59}$$

In (59) we have written

$$(y) = \sum \frac{ye_n^2}{y+1/\lambda_n} = \sum \frac{y\lambda_n e_n^2}{1+y\lambda_n} \tag{60}$$

and y_0 is determined by

$$f'(y_0) = 0. \tag{61}$$

We will pay particular attention to the function $f(y)$, since it is this function which determines the exponential behavior for large S/N. We note that in terms of the system operator G (60), which defines the function $f(y)$, may be written

$$f(y) = \left(s, \frac{yG}{1+yG} s\right) \tag{62}$$

where we use the usual notation (a,b) for the inner product of two vectors in Hilbert space. In (62) we have used s to denote the vector $s(t-v)$, t fixed. If one knew explicitly the resolvent operator $(1+yG)^{-1}$ appearing in (62), one might perhaps calculate the required integrals in (62) and search for the maximum of this function of y, thus determining y_0 which is implicitly a functional of the signal s. In general, however, approximation methods must be used.

Before mentioning some approximation schemes, we would like to demonstrate an interesting result. The saddle point y_0 is determined by (61), or, using (62), we have the implicit relationship

$$y_0 = \frac{\left(s, \frac{G}{1+y_0G} s\right)}{\left(s, \frac{G^2}{(1+y_0G)^2} s\right)}. \tag{63}$$

Thus, from (62) and (63),

$$f(y_0) = y_0^2 \left(s, \frac{G^2}{(1+y_0G)^2} s\right), \tag{64}$$

or

$$f(y_0) = \frac{(s,Ks)^2}{(s,K^2s)}, \tag{65}$$

where

$$K = \frac{G}{1 + y_0G}. \tag{66}$$

Now clearly,

$$f(y_0) = \frac{(s,Ks)^2}{(s,K^2s)} \leqq \frac{(s,s)(Ks,Ks)}{(Ks,Ks)} = (s,s) \tag{67}$$

by hermiticity of K and Schwarz's inequality. Thus, if the signal energy $E = (s,s)$ is fixed, the best performance would be achieved if $(1/E)f(y_0) = 1$, provided that this is possible. Certainly the equality in (67) is satisfied if s is an eigenfunction of K, and hence of the system function G, and at first glance this would appear to be an optimizing solution. However, such a solution violates the necessary condition that the function $f(y)$ have poles at the nearest eigenvalue for both positive and negative y, and hence our basic relation (65) does not hold for such a choice of s. In fact, under the assumptions for which (65) was derived, no function s, subject to the constant energy constraint, will yield a stationary value for the exponent (65).* In fact, it is not difficult to convince oneself that the best function s to take (for a positive output) is the eigenfunction φ_0 with the largest positive eigenvalue. Note that (65) is not applicable here. However, the exponent may be estimated according to the discussion of Section III to be

$$f\left(y = \left|\frac{1}{\lambda_-}\right|\right) = e_1^2 \frac{\left|\frac{\lambda_+}{\lambda_-}\right|}{1 + \left|\frac{\lambda_+}{\lambda_-}\right|} \tag{68}$$

where λ_+ (λ_-) is the positive (negative) eigenvalue of G with largest magnitude. If λ_+ is large compared with $|\lambda_-|$, the equality in (67) may be approached (note $e_1^2 = (s,s)$) for positive pulses. However, for negative pulses the factor in the exponent will be

$$\frac{\left|\frac{\lambda_-}{\lambda_+}\right|}{1 + \left|\frac{\lambda_-}{\lambda_+}\right|} \tag{69}$$

* Even though the exponent is a function of y_0 which implicitly depends on s this dependence can be ignored in a first-order variation because of (61).

and will become arbitrarily bad. Thus, if one wants symmetry between positive and negative pulses, one must take λ_+ and λ_- to have essentially the same magnitude, leading to a result that is a factor of two worse than suggested by the equality in (67). We might point out that the value of the exponent that would be obtained if the equality in (67) held, is the same as that for the probability of error for an ideal correlator, the optimum detector for tbis binary situation. Since, as just described, the optimum binary scheme here uses orthogonal signals, the factor of two worse than ideal binary correlation detection is not surprising. Since the ideal binary system (again a correlator) for *orthogonal* signals also gives an exponent worse by a factor of two, no exponent could be better.

VI. EXAMPLE OF QUADRATIC DETECTOR

A particular specialization of the general quadratic detector given schematically in Fig. 1 is a differential detector. By this term we shall mean that the filters F_1 and F_2 in Fig. 1 are identical except that F_2, in addition to representing the channel as F_1 does, has a delay of one bit interval. The filter F_3 is a low-pass filter. We treat the simplified baseband case where F_1 is an ideal low-pass filter with cutoff Ω rad/s., and F_2 is identical to F_1 except for a delay T_1, and finally the postdetection filter F_3 is an ideal integrator with integration time T seconds. The analysis is also relevant to the situation where F_1 and F_2 are bandpass filters, symmetrical with respect to some carrier frequency, provided one neglects the double carrier-frequency terms at the input to F_3.

We begin by considering first the alternative version of the output (51), namely

$$E_{\text{out}}(t) = \tfrac{1}{2}[(s_d, \bar{g}s) + (s, \bar{g}s_d)], \tag{70}$$

where in (70) we have for convenience included tbe delay T_1 directly in the signal rather than in the system function, and have denoted the delayed version of s by s_d. In (70), $\bar{g}$ denotes the system function for two identical filters F_1 and F_2. Equation (49) now reads

$$\bar{g}(u,v) = \int_{-T/2}^{T/2} \frac{\sin \Omega\ (u - z)}{\pi(u - z)} \frac{\sin \Omega\ (v - z)}{\pi(v - z)}\, dz. \tag{71}$$

We would like to evaluate the integral for u and v on the entire real line. We shall first present a formal evaluation for u and v both restricted to the interval $(-T/2, T/2)$, and shall then invoke analytic continuation to claim that this restriction on the evaluation of (71) can be dropped. Let $\psi_n(t)$ be the prolate spheroidal function normalized to unity in the

infinite interval, and hence to λ_n on the interval $(-T/2, T/2)$. Also let $\psi_n'(t)$ be the same function normalized to unity on the interval $(-T/2, T/2)$ and hence to $1/\lambda_n$ on the infinite interval. The functions $\psi_n'(t)$ satisfy the equation[14]

$$\int_{-T/2}^{T/2} \frac{\sin \Omega\,(u - z)}{\pi(u - z)}\, \psi_n'(z)\, dz = \lambda_n \psi_n'(u), \tag{72}$$

where the eigenvalues λ_n are real and positive, and $\psi_n(z)$ satisfy the identical equation. On the Hilbert space of square-integrable functions on the interval $(-T/2, T/2)$, it then follows that we may write

$$\frac{\sin \Omega\,(u - z)}{\pi(u - z)} = \sum_{n=0}^{\infty} \lambda_n \psi_n'(u) \psi_n'(z). \tag{73}$$

Therefore,

$$\bar{g}(u,v) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \lambda_n \lambda_m \psi_n'(u) \psi_m'(v) \int_{-T/2}^{T/2} \psi_n'(z) \psi_m'(z)\, dz,$$

or

$$\bar{g}(u,v) = \sum_{n=0}^{\infty} \lambda_n^2 \psi_n'(u) \psi_n'(v). \tag{74}$$

Reinterpreting (74) to hold on the infinite interval, we have finally

$$\bar{g}(u,v) = \sum_{n=0}^{\infty} \lambda_n \psi_n(u) \psi_n(v). \tag{75}$$

Rewriting (70) as

$$E_{\text{out}} = \tfrac{1}{2}[(s, U^+(T_1)\bar{g}s) + (s, \bar{g}U(T_1)s)], \tag{76}$$

where $U(T_1)$ represents the unitary operator of time translation by amount T_1, and comparing (76) with (51), we see that the symmetrized system function $G(u,v)$ is given by

$$\begin{aligned} G(u,v) = \tfrac{1}{2} \sum_n \lambda_n \psi_n(u + T_1) \psi_n(v) \\ + \tfrac{1}{2} \sum_n \lambda_n \psi_n(u) \psi_n(v + T_1). \end{aligned} \tag{77}$$

Important simplifications in the result (77) obtain when one makes use of the fact that the λ_n tend to zero rapidly after a few terms. Thus, one would expect that the infinite sum for the system function (77) may be effectively truncated after a few terms, and the problem of finding the resolvent kernel $(1 + yG)^{-1}$ in (62) is thereby reduced to inverting a finite dimensional matrix. The rapidity with which the λ_n decrease

depends on the parameter[14] $c = \Omega T/2$. Some fairly typical situations correspond roughly to $c = \pi$ and $T_1 = T$. The first six eigenvalues for $c = \pi$, which are 0.981, 0.749, 0.243, 0.025, 0.001, $\sim 10^{-5}$, illustrate this behavior well. Furthermore, it usually happens that the shape of the signal during the integration time of the filter, i.e., the pulse shape, bears a great deal of resemblance to $\psi_0(t)$, which one may crudely visualize as having a $(\sin t)/t$ shape. This tends to emphasize the $n = 0$ term in (77) even more. Hence we should not be far wrong if we simply write (for $T_1 = T$)

$$G(u,v) = \tfrac{1}{2}\lambda_0\psi_0(u + T)\psi_0(v) + \tfrac{1}{2}\lambda_0\psi_0(u)\psi_0(v + T), \tag{78}$$

or equivalently if we define

$$\epsilon = \int_{-T/2}^{T/2} \psi_0(u)\psi_0(u + T)\, du, \tag{79}$$

we have that

$$\begin{aligned} G(u,v) = {} & \frac{\lambda_0(1 + \epsilon)}{2}\left[\frac{\psi_0(u) + \psi_0(u + T)}{\sqrt{2(1 + \epsilon)}}\right]\left[\frac{\psi_0(v) + \psi_0(v + T)}{\sqrt{2(1 + \epsilon)}}\right] \\ & - \frac{\lambda_0(1 - \epsilon)}{2}\left[\frac{\psi_0(u) - \psi_0(u + T)}{\sqrt{2(1 - \epsilon)}}\right]\left[\frac{\psi_0(v) - \psi_0(v + T)}{\sqrt{2(1 - \epsilon)}}\right]. \end{aligned} \tag{80}$$

The latter form (80) explicitly exhibits the eigenvalues and eigenfunctions to this approximation. For $T_1 = T$, ϵ is quite small and in the spirit of our approximation may be neglected. We note in closing that we have found the representations (78) through (80) extremely useful in evaluating the effects of an added external tone on differential phase detection of a signal accompanied by gaussian noise. We emphasize that the dependence of the degradation on the frequency of the tone in such a problem is strongly influenced by the presence of the postdetection filter and hence its inclusion (aside from its role of selecting only difference frequencies) was essential. This, in fact, motivated much of the present work.

APPENDIX

Proof of Monotonicity of Exponent on the Contour

The function of interest is

$$\exp\,[-f(\omega)/\sigma^2] = \exp\left[-\frac{\omega}{2\sigma^2}\sum_{j=1}^{N}\frac{y_j^2}{\omega + \dfrac{i}{d_j}}\right]. \tag{81}$$

If we let $\omega = x + i\Gamma$ with x real and with the saddle point occurring at $\omega = i\Gamma$, we have on the contour L_2

$$|\exp\{-f(\omega)/\sigma^2\}| = \exp\left\{-\frac{1}{2\sigma^2}\sum_j \frac{y_j^2\left[x^2 + \Gamma\left(\Gamma + \frac{1}{d_j}\right)\right]}{x^2 + \left(\Gamma + \frac{1}{d_j}\right)^2}\right\}. \quad (82)$$

Therefore, it is sufficient for us to prove for those j such that $y_j^2 \neq 0$, that

$$\frac{x^2 + \Gamma\left(\Gamma + \frac{1}{d_j}\right)}{x^2 + \left(\Gamma + \frac{1}{d_j}\right)^2} > \frac{\Gamma\left(\Gamma + \frac{1}{d_j}\right)}{\left(\Gamma + \frac{1}{d_j}\right)^2}, \quad (83)$$

which in turn amounts to showing

$$\frac{\frac{1}{d_j}}{\Gamma + \frac{1}{d_j}} > 0. \quad (84)$$

Now, if we merely recall that $\Gamma > 0$, then the inequality is obviously true for $d_j > 0$. If we also recall that if $d_j < 0$ then $(-1)/d_j > \Gamma$, (84) obviously holds for $d_j < 0$.

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